

CS-184: Computer Graphics

Lecture #6: 3D Transformations and Rotations

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Today

- Transformations in 3D
- Rotations
 - Matrices
 - Euler angles
 - Exponential maps
 - Quaternions

3D Transformations

- Generally, the extension from 2D to 3D is straightforward
 - Vectors get longer by one
 - Matrices get extra column and row
 - SVD still works the same way
 - Scale, Translation, and Shear all basically the same
- Rotations get interesting

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Translations

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{For 2D}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{For 3D}$$

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Scales

$$\tilde{\mathbf{A}} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For 2D

$$\tilde{\mathbf{A}} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3D

(Axis-aligned!)

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Shears

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For 2D

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3D

(Axis-aligned!)

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Shears

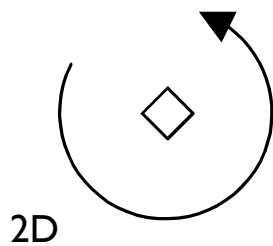
$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shears y into x

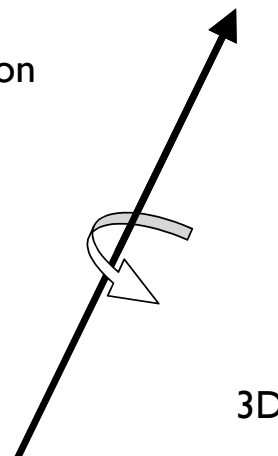
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Rotations

- 3D Rotations fundamentally more complex than in 2D
 - 2D: amount of rotation
 - 3D: amount and axis of rotation



-VS-



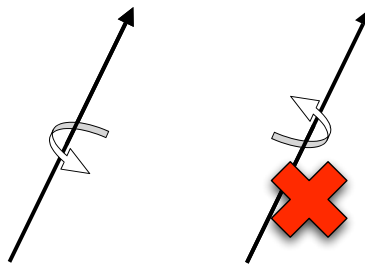
2D

3D

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Rotations

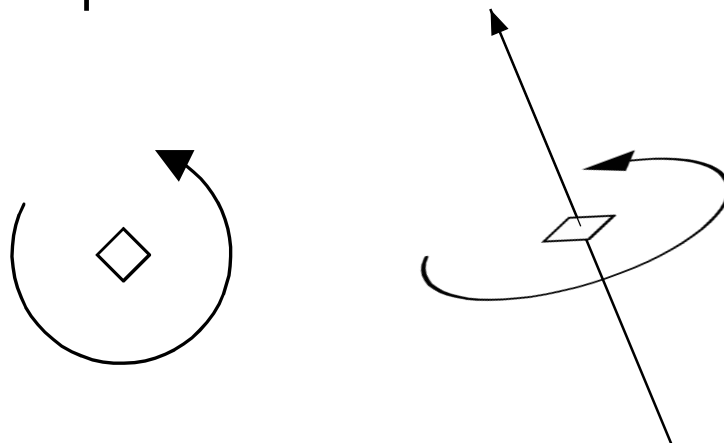
- Rotations still orthonormal
- $\text{Det}(\mathbf{R}) = 1 \neq -1$
- Preserve lengths and distance to origin
- 3D rotations **DO NOT COMMUTE!**
- Right-hand rule
- Unique matrices



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Axis-aligned 3D Rotations

- 2D rotations implicitly rotate about a third out of plane axis

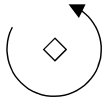


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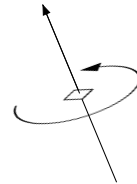
Axis-aligned 3D Rotations

- 2D rotations implicitly rotate about a third out of plane axis

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Note: looks same as $\tilde{\mathbf{R}}$



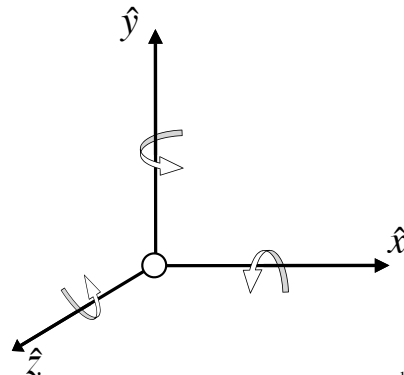
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Axis-aligned 3D Rotations

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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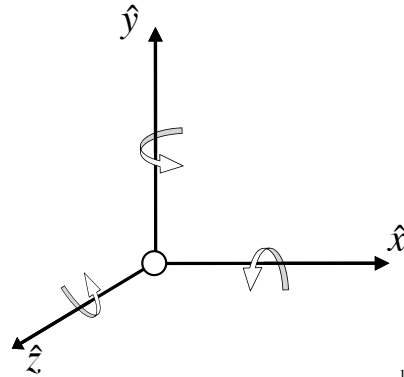
Axis-aligned 3D Rotations

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

“Z is in your face”



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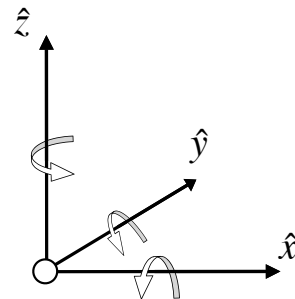
Axis-aligned 3D Rotations

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also right handed “Zup”



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Axis-aligned 3D Rotations

- Also known as “direction-cosine” matrices

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

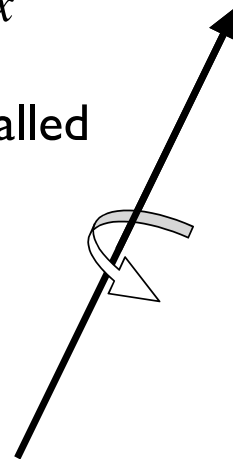
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Arbitrary Rotations

- Can be built from axis-aligned matrices:

$$\mathbf{R} = \mathbf{R}_{\hat{z}} \cdot \mathbf{R}_{\hat{y}} \cdot \mathbf{R}_{\hat{x}}$$

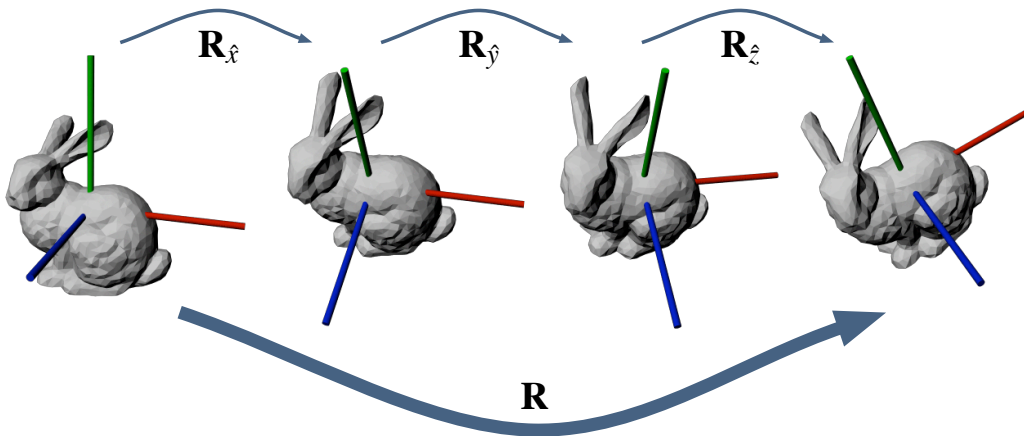
- Result due to Euler... hence called Euler Angles
- Easy to store in vector
- But NOT a vector.



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Arbitrary Rotations

$$\mathbf{R} = \mathbf{R}_{\hat{z}} \cdot \mathbf{R}_{\hat{y}} \cdot \mathbf{R}_{\hat{x}}$$



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Arbitrary Rotations

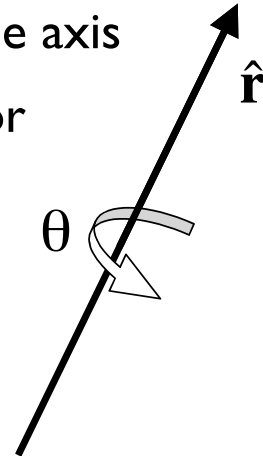
- Allows tumbling
- Euler angles are non-unique
- Gimbal-lock
- Moving -vs- fixed axes
 - Reverse of each other

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Exponential Maps

- Direct representation of arbitrary rotation
- AKA: axis-angle, angular displacement vector
- Rotate θ degrees about some axis
- Encode θ by length of vector

$$\theta = |\mathbf{r}|$$



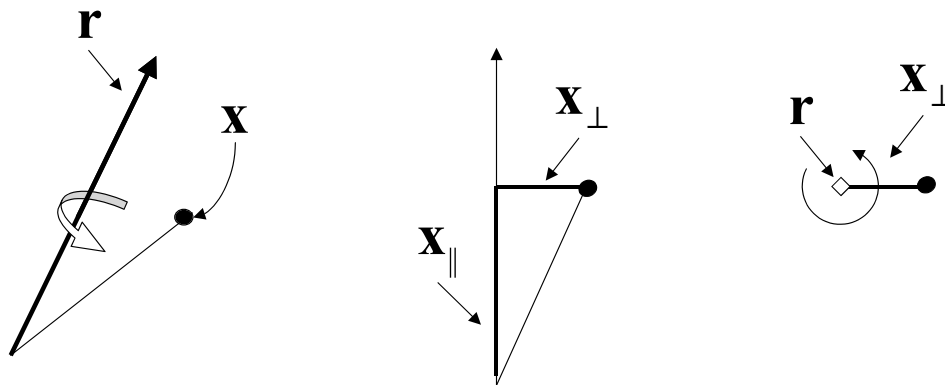
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Exponential Maps

- Given vector \mathbf{r} , how to get matrix \mathbf{R}
- Method from text:
 1. rotate about x axis to put \mathbf{r} into the x - y plane
 2. rotate about z axis align \mathbf{r} with the x axis
 3. rotate θ degrees about x axis
 4. undo #2 and then #1
 5. composite together

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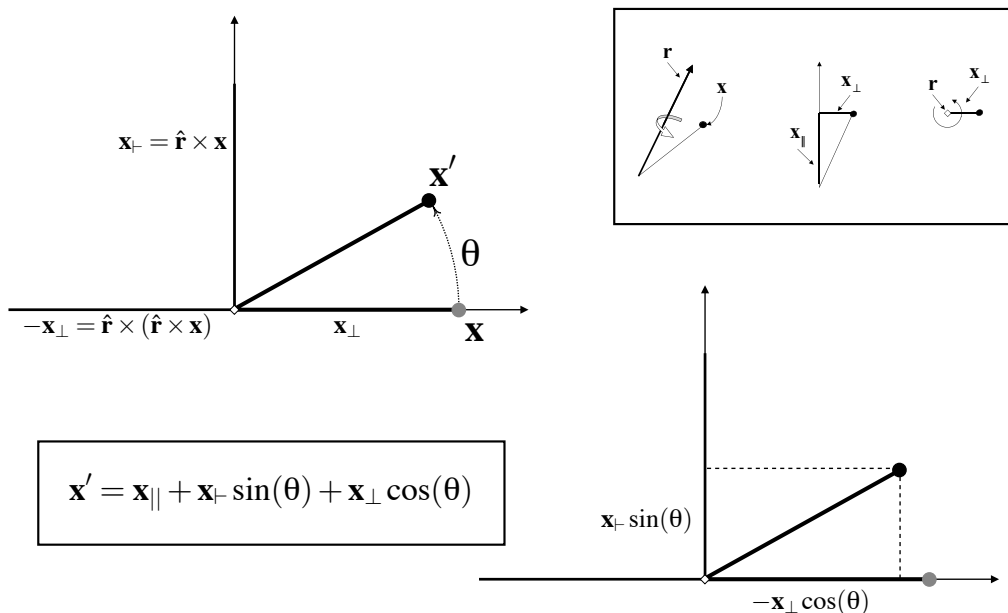
Exponential Maps



- Vector expressing a point has two parts
 - \mathbf{x}_{\parallel} does not change
 - \mathbf{x}_{\perp} rotates like a 2D point

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Exponential Maps

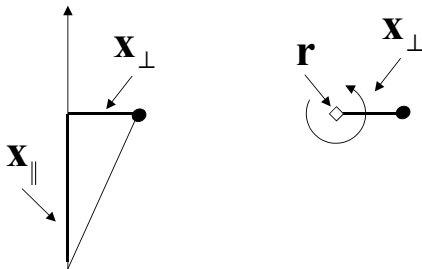


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Exponential Maps

- Rodriguez Formula

$$\begin{aligned} \mathbf{x}' &= \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{x}) \\ &+ \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{x}) \\ &- \cos(\theta)(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{x})) \end{aligned}$$



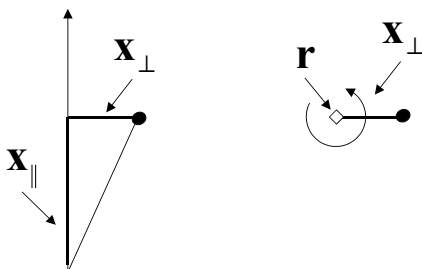
Actually a minor variation ... 22

Exponential Maps

- Rodriguez Formula

$$\begin{aligned} \mathbf{x}' &= \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{x}) \\ &+ \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{x}) \\ &- \cos(\theta)(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{x})) \end{aligned}$$

Linear in \mathbf{x}



Actually a minor variation ... 22

Exponential Maps

- Building the matrix

$$\mathbf{x}' = ((\hat{\mathbf{r}}\hat{\mathbf{r}}^t) + \sin(\theta)(\hat{\mathbf{r}}\times) - \cos(\theta)(\hat{\mathbf{r}}\times)(\hat{\mathbf{r}}\times)) \mathbf{x}$$

$$(\hat{\mathbf{r}}\times) = \begin{bmatrix} 0 & -\hat{r}_z & \hat{r}_y \\ \hat{r}_z & 0 & -\hat{r}_x \\ -\hat{r}_y & \hat{r}_x & 0 \end{bmatrix}$$

Antisymmetric matrix

$$(\mathbf{a}\times)\mathbf{b} = \mathbf{a}\times\mathbf{b}$$

Easy to verify by expansion

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Exponential Maps

- Allows tumbling
- No gimbal-lock!
- Orientations are space within π -radius ball
- Nearly unique representation
- Singularities on shells at 2π
- Nice for interpolation

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Exponential Maps

- Why exponential?
- Recall series expansion of e^x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

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Exponential Maps

- Why exponential?
- Recall series expansion of e^x
- Euler: what happens if you put in $i\theta$ for x

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= \left(1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \right) + i \left(\frac{\theta}{1!} + \frac{-\theta^3}{3!} + \dots \right) \\ &= \cos(\theta) + i \sin(\theta) \end{aligned}$$

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Exponential Maps

- Why exponential?

$$e^{(\hat{\mathbf{r}} \times) \theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}} \times) \theta}{1!} + \frac{(\hat{\mathbf{r}} \times)^2 \theta^2}{2!} + \frac{(\hat{\mathbf{r}} \times)^3 \theta^3}{3!} + \frac{(\hat{\mathbf{r}} \times)^4 \theta^4}{4!} + \dots$$

But notice that: $(\hat{\mathbf{r}} \times)^3 = -(\hat{\mathbf{r}} \times)$

$$e^{(\hat{\mathbf{r}} \times) \theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}} \times) \theta}{1!} + \frac{(\hat{\mathbf{r}} \times)^2 \theta^2}{2!} + \frac{-(\hat{\mathbf{r}} \times) \theta^3}{3!} + \frac{-(\hat{\mathbf{r}} \times)^2 \theta^4}{4!} + \dots$$

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Exponential Maps

$$e^{(\hat{\mathbf{r}} \times) \theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}} \times) \theta}{1!} + \frac{(\hat{\mathbf{r}} \times)^2 \theta^2}{2!} + \frac{-(\hat{\mathbf{r}} \times) \theta^3}{3!} + \frac{-(\hat{\mathbf{r}} \times)^2 \theta^4}{4!} + \dots$$

$$e^{(\hat{\mathbf{r}} \times) \theta} = (\hat{\mathbf{r}} \times) \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \dots \right) + \mathbf{I} + (\hat{\mathbf{r}} \times)^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots \right)$$

$$e^{(\hat{\mathbf{r}} \times) \theta} = (\hat{\mathbf{r}} \times) \sin(\theta) + \mathbf{I} + (\hat{\mathbf{r}} \times)^2 (1 - \cos(\theta))$$

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Quaternions

- More popular than exponential maps
- Natural extension of $e^{i\theta} = \cos(\theta) + i \sin(\theta)$
- Due to Hamilton (1843)
 - Interesting history
 - Involves “hermaphroditic monsters”

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Quaternions

- Uber-Complex Numbers

$$q = (z_1, z_2, z_3, s) = (\mathbf{z}, s)$$

$$q = iz_1 + jz_2 + kz_3 + s$$

$$i^2 = j^2 = k^2 = -1$$
$$\begin{array}{ll} ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{array}$$

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Quaternions

- Multiplication natural consequence of defn.

$$q \cdot p = (\mathbf{z}_q s_p + \mathbf{z}_p s_q + \mathbf{z}_p \times \mathbf{z}_q, s_p s_q - \mathbf{z}_p \cdot \mathbf{z}_q)$$

- Conjugate

$$q^* = (-\mathbf{z}, s)$$

- Magnitude

$$\|q\|^2 = \mathbf{z} \cdot \mathbf{z} + s^2 = q \cdot q^*$$

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Quaternions

- Vectors as quaternions

$$v = (\mathbf{v}, 0)$$

- Rotations as quaternions

$$r = (\hat{\mathbf{r}} \sin \frac{\theta}{2}, \cos \frac{\theta}{2})$$

- Rotating a vector

$$x' = r \cdot x \cdot r^* \leftarrow \text{Compare to Exp. Map}$$

- Composing rotations

$$r = r_1 \cdot r_2$$

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Quaternions

- No tumbling
- No gimbal-lock
- Orientations are “double unique”
- Surface of a 3-sphere in 4D $\|r\| = 1$
- Nice for interpolation

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Rotation Matrices

- Eigen system
 - One real eigenvalue
 - Real axis is axis of rotation
 - Imaginary values are 2D rotation as complex number
- Logarithmic formula

$$(\hat{\mathbf{r}} \times) = \ln(\mathbf{R}) = \frac{\theta}{2 \sin \theta} (\mathbf{R} - \mathbf{R}^T)$$

$$\theta = \cos^{-1} \left(\frac{\text{Tr}(\mathbf{R}) - 1}{2} \right)$$

Similar formulae as for exponential...³⁴

Rotation Matrices

- Consider:

$$\mathbf{RI} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Columns are coordinate axes after transformation (true for general matrices)
- Rows are original axes in original system (not true for general matrices)

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Note:

- Rotation stuff in the book is a bit weak... luckily you have these nice slides!

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