Outline

- Last time:
  - Field-Programmable Logic Arrays
  - FPGA-Based FSM Design
  - Intro. to the Xilinx FPGA
- This lecture:
  - Cube Notation
  - Postulates and Theorems of Boolean Algebra
  - De Morgan's Laws
  - Algebraic Simplification
  - Standard Forms, Min-terms, Short Form
  - Design Example: Simplify algebraically
  - Define implicant, prime implicant, cover, prime cover

Design Example: Seatbelt

Define variables:
S = car starts, B = belts are buckled, D = Door is closed, P = brake is on

B	D	P	S′	S
0	0	0	1	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

S′ = B′D′P′ + B′DP′ + B′DP
+ BD′P′ + BD′P + BDP

S = B′D′P + BDP′

Standard Forms: Definitions

A *literal* is a variable *or* the complement of a variable. (e.g. X, Y, X')

A *product term* is a single literal or the logical product of two or more literals. (e.g. X•Y, W•X'•Z, X)

A *sum term* is a single literal or the logical sum of two or more literals. (e.g. X+Y, X, W+X'+Y)

A *sum-of-products* expression is a logical sum of product terms. (e.g. X•Y + W'•Y'•Z)

A *product-of-sums* expression is a logical product of sum terms. (e.g. (X+Y)•(Z+W+V))

A *normal term* is a product or sum term in which no variable appears more than once.

An n-variable *minterm* is a normal product term with n literals. (e.g. X•Y•Z, W•X'•Y)

An n-variable *maxterm* is a normal sum term with n literals. (e.g. W'+X'+Y, X+Y+Z)

Minterms and Maxterms

- Use the *ones* of the function ("on-set") to determine *minterms* and the *zeros* of the function to determine *maxterm* form.

X	Y	Z
0	**0**	**0**
0	**1**	**1**
1	**0**	**1**
1	**1**	**0**

XOR

$Z(X,Y) = X'\bullet Y + X\bullet Y'$
$= \Sigma m(1,2)$

$Z'(X,Y) = X'\bullet Y' + X\bullet Y$
$Z(X,Y) = (X'\bullet Y' + X\bullet Y)'$
$= (X+Y)\bullet(X'+Y')$
$= \Pi M(0,3)$

Standard Forms

	B	D	P	S
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	0
6	1	1	0	1
7	1	1	1	0

$S(B,D,P) = \Sigma m(\quad)$

$S(B,D,P) = \Pi M(\quad)$

- **The notation above is called the *short-form representation* of the Sum-of-Products and Product-of-Sums *canonical forms* respectively.**

De Morgan's Laws

Given a switching function f(X,Y,...), to obtain its complement f':

(1) Change all + operations to •.

(2) Change all • operations to +.

(3) Complement each variable

e.g. $f'(X,Y,Z) = X + X\bullet Y'\bullet Z'\bullet(X\bullet Y + Z\bullet Y')$

$f(X,Y,Z) = X'\bullet(X'+Y+Z) + (X'+Y')\bullet(Z'+Y)$

Product of Sums: Conjunctive Normal Form

A	B	C	Maxterms
0	0	0	$A+B+C = M_0$
0	0	1	$A+B+\overline{C} = M_1$
0	1	0	$A+\overline{B}+C = M_2$
0	1	1	$A+\overline{B}+\overline{C} = M_3$
1	0	0	$\overline{A}+B+C = M_4$
1	0	1	$\overline{A}+B+\overline{C} = M_5$
1	1	0	$\overline{A}+\overline{B}+C = M_6$
1	1	1	$\overline{A}+\overline{B}+\overline{C} = M_7$

Maxterm Shorthand Notation for a Function of Three Variables

→ *Maxterm:*

ORed sum of literals in which each variable appears exactly once in either true or complemented form, but not both!

→ *Maxterm form:*

Find truth table rows where *f* is 0
0 in input column implies true literal
1 in input column implies complemented literal

***f*(A,B,C) = ΠM(0,1,2)**
= (A + B + C) (A + B + C') (A + B' + C)
***f'*(A,B,C) = ΠM(3,4,5,6,7)**
= (A + B' + C) (A' + B + C) (A' + B + C') (A' + B' + C) (A' + B' + C')

Design Example: The Hot-tub

George is concerned as to whether he should warm up the hot-tub for tonight's party. Under what conditions will someone use the tub?:

(a) Alice never bathes

(b) Bill bathes, iff both Alice and Don are present

(c) Cindy bathes under all conditions - even by herself!

(d) Don bathes iff Alice is not present.

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A = Alice is present, B = Bill is present, C = Cindy is present, D = Don is present, T = someone will use the tub.

T = BAD + C + DA'

= ?

Word Statement: The hot-tub will be used if

Fundamental Theorems & Postulates of Boolean Algebra

Identities:	**(1)**	**X + 0 = X**	**(1D)**	**X • 1 = X**
Null Elements:	**(2)**	**X + 1 = 1**	**(2D)**	**X • 0 = 0**
Indempotency:	**(3)**	**X + X = X**	**(3D)**	**X • X = X**
Involution (Double Negation):			**(4)**	**(X')' = X**
Complements:	**(5)**	**X + X' = 1**	**(5D)**	**X • X' = 0**
Commutativity:	**(6)**	**X + Y = Y + X**	**(6D)**	**X • Y = Y • X**
Associativity:	**(7)**	**(X+Y)+Z = X+(Y+Z)**	**(7D)**	**(X • Y) • Z = X • (Y • Z)**
Distributivity:	**(8)**	**X • Y+X • Z = X • (Y+Z)**	**(8D)**	**(X+Y) • (X+Z) = X+Y • Z**
Combining:	**(9)**	**X • Y + X • Y' = X**	**(9D)**	**(X+Y) • (X+Y') = X**
Covering:	**(10)**	**X + X • Y = X**	**(10D)**	**X • (X+Y) = X**
DeMorgan's Laws:	**(12)**	**(X • Y • Z)' = X'+Y'+Z'**	**(12D)**	**(X+Y+Z)' = X' • Y' • Z'**
Consensus:	**(17)**	**X • Y+X' • Z+Y • Z = X • Y+X' • Z**	**(17D)**	**(X+Y) • (X'+Z) • (Y+Z) = (X+Y) • (X'+Z)**
Shannon Expansion:	**(18)**	**F(X,Y,Z) = X • F(1,Y,Z) + X' • F(0,Y,Z)**		
	(18D)	**F(X,Y,Z) = (X+F(0,Y,Z)) • (X'+F(1,Y,Z))**		

Design Example: The Hot-tub

Reduce using fundamental theorems & postulates (label equation on right side from table)

T = BAD + C + DA'

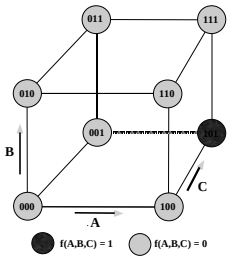
=

=

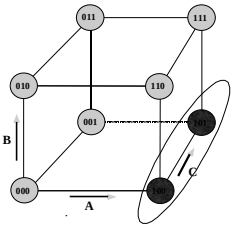
=

Word statement: The tub should be warmed up iff:

Cubes as Product Terms

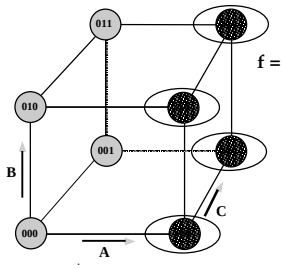


f = AB'C



f = AB'C + AB'C'
= AB'

f (A,B,C) = A
Using Minterms (4 cubes)



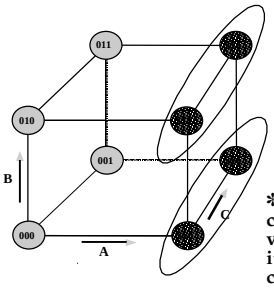
f = AB'C' + AB'C + ABC' + ABC

Alternate format:

A	B	C	f
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

ON-set or ON-cover

f(A,B,C) = A
Two cubes



f = AB + AB'

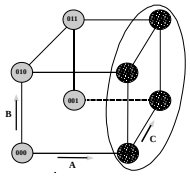
Alternate format:

A	B	C	f
1	1	*	1
1	0	*	1

Input "don't cares"

*** = "don't care": The cube is independent of that variable; we "don't care" what it's value is - it doesn't matter c.f. PLA "don't connects" from last week!**

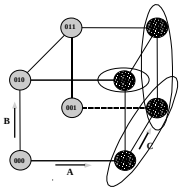
f(A,B,C) = A
One Cube and Mixed Cover



f = A

Alternate format:

A	B	C	f
1	*	*	1

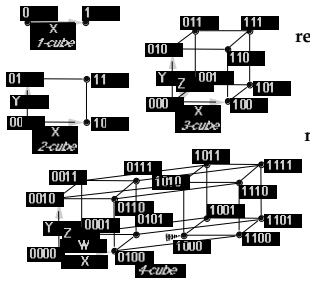


f = ?

Alternate format:

A	B	C	f
1	1	0	1
1	0	*	1
1	*	1	1

Boolean Hypercubes

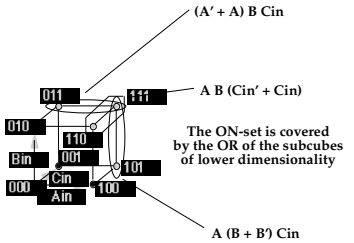


Just another way to represent the truth table

n input variables = n dimensional "cube"

Relationship to Cube Notation:
Full Adder

Ain	Bin	Cin	Cout
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



(A' + A) B Cin

A B (Cin' + Cin)

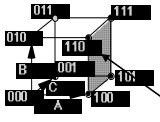
The ON-set is covered by the OR of the subcubes of lower dimensionality

A (B + B') Cin

Cout = B Cin + A B + A Cin

Sub-cubes of Higher Dimensionality

***f*(A,B,C) = Σm(4,5,6,7)**



On-set forms a rectangle, i.e., a cube of two dimensions

represents an expression in one variable i.e., 3 dimensions - 2 dimensions

A is asserted and unchanged B and C vary

This subcube represents the literal A

Describing Logic Functions

❍**Definitions:**

- ➜ Adjacent pairs, quadruples, octuples, ... of minterms (a vertex marked 1) in the Boolean space are called *implicants.*
- ➜ An implicant is called a *prime implicant* if it is not a subset of another implicant of the function.
- ➜ Any prime implicant which includes a 1 vertex that is not included in any other prime implicant is called an *essential prime implicant*.
- ➜ A set of prime implicants of a function is called a *minimal sum-of-products form (minimal "cover")* of the function if the following conditions are satisfied:
  - (a) It includes the entire on-set (1 verticies) of the function.
  - (b) Condition (a) will not be satisfied if any one of the prime implicants is removed.
  - (c) None of the prime implicants can be replaced with a simpler prime implicant (i.e. one containing fewer variables).