Outline

- Last time:
  - Field-Programmable Logic Arrays
  - FPGA-Based FSM Design
  - Intro. to the Xilinx FPGA
- This lecture:
  - Cube Notation
  - Postulates and Theorems of Boolean Algebra
  - De Morgan's Laws
  - Algebraic Simplification
  - Standard Forms, Min-terms, Short Form
  - Design Example: Simplify algebraically
  - Define implicant, prime implicant, cover, prime cover

Design Example: Seatbelt

Define variables:
S = car starts, B = belts are buckled, D = Door is closed, P = brake is on

B	D	P	S'	S
0	0	0	1	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

S' = B'D'P' + B'DP' + B'DP
+ BD'P' + BD'P + BDP

S = B'D'P + BDP'

Standard Forms: Definitions

A *literal* is a variable *or* the complement of a variable. (e.g. X, Y, X')

A *product term* is a single literal or the logical product of two or more literals. (e.g. X•Y, W•X'•Z, X)

A *sum term* is a single literal or the logical sum of two or more literals. (e.g. X+Y, X, W+X'+Y)

A *sum-of-products* expression is a logical sum of product terms. (e.g. X•Y + W'•Y'•Z)

A *product-of-sums* expression is a logical product of sum terms. (e.g. (X+Y)•(Z+W+V))

A *normal term* is a product or sum term in which no variable appears more than once.

An n-variable *minterm* is a normal product term with n literals. (e.g. X•Y•Z, W•X'•Y)

An n-variable *maxterm* is a normal sum term with n literals. (e.g. W'+X'+Y, X+Y+Z)

Minterms and Maxterms

- Use the *ones* of the function ("on-set") to determine *minterms* and the *zeros* of the function to determine *maxterm* form.

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

XOR

$$Z(X,Y) = X' \cdot Y + X \cdot Y'$$
$$= \Sigma m(1,2)$$

$$Z'(X,Y) = X' \cdot Y' + X \cdot Y$$
$$Z(X,Y) = (X' \cdot Y' + X \cdot Y)'$$
$$= (X+Y) \cdot (X'+Y')$$
$$= \Pi M(0,3)$$

Standard Forms

	B	D	P	S
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	0
6	1	1	0	1
7	1	1	1	0

$S(B,D,P) = \Sigma m(\quad)$

$S(B,D,P) = \Pi M(\quad)$

➔ **The notation above is called the *short-form representation* of the Sum-of-Products and Product-of-Sums *canonical forms* respectively.**

De Morgan's Laws

Given a switching function f(X,Y,...), to obtain its complement f':

(1) Change all + operations to •.

(2) Change all • operations to +.

(3) Complement each variable

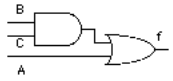
e.g. $f'(X,Y,Z) = X + X \bullet Y' \bullet Z' \bullet (X \bullet Y + Z \bullet Y')$

$f(X,Y,Z) = X' \bullet (X'+Y+Z) + (X'+Y') \bullet (Z'+Y)$

Sum of Products: Disjunctive Normal Form

A	B	C	Minterms
0	0	0	$\bar{A}\bar{B}\bar{C} = m_0$
0	0	1	$\bar{A}\bar{B}C = m_1$
0	1	0	$\bar{A}B\bar{C} = m_2$
0	1	1	$\bar{A}BC = m_3$
1	0	0	$A\bar{B}\bar{C} = m_4$
1	0	1	$A\bar{B}C = m_5$
1	1	0	$AB\bar{C} = m_6$
1	1	1	$ABC = m_7$

Shorthand Notation for Minterms of 3 Variables



2-Level AND/OR Realization

➔ *product term / minterm:*

ANDed product of literals in which each variable appears exactly once, in true or complemented form (but not both!)

➔ *f in canonical form:*

$f(A,B,C) = \Sigma m(3,4,5,6,7)$
$= m3 + m4 + m5 + m6 + m7$
$= A'\,B\,C + A\,B'\,C' + A\,B'\,C + A\,B\,C' + A\,B\,C$

➔ *canonical form ° minimal form*

$f = A\,B'\,(C + C') + A'\,B\,C + A\,B\,(C' + C)$
$= A\,B' + A'\,B\,C + A\,B$
$= A\,(B' + B) + A'\,B\,C$
$= A + A'\,B\,C$
$= A + B\,C$

$f' = (A + B\,C)' = A'\,(B' + C') = A'\,B' + A'\,C'$

Product of Sums: Conjunctive Normal Form

A	B	C	Maxterms
0	0	0	$A+B+C = M_0$
0	0	1	$A+B+\bar{C} = M_1$
0	1	0	$A+\bar{B}+C = M_2$
0	1	1	$A+\bar{B}+\bar{C} = M_3$
1	0	0	$\bar{A}+B+C = M_4$
1	0	1	$\bar{A}+B+\bar{C} = M_5$
1	1	0	$\bar{A}+\bar{B}+C = M_6$
1	1	1	$\bar{A}+\bar{B}+\bar{C} = M_7$

Maxterm Shorthand Notation for a Function of Three Variables

➔ *Maxterm:*

ORed sum of literals in which each variable appears exactly once in either true or complemented form, but not both!

➔ *Maxterm form:*

Find truth table rows where f is 0
0 in input column implies true literal
1 in input column implies complemented literal

$f(A,B,C) = \Pi M(0,1,2)$
$= (A + B + C)\,(A + B + C')\,(A + B' + C)$

$f'(A,B,C) = \Pi M(3,4,5,6,7)$
$= (A + B' + C)\,(A' + B + C)\,(A' + B + C')\,(A' + B' + C)\,(A' + B' + C')$

Design Example: The Hot-tub

George is concerned as to whether he should warm up the hot-tub for tonight's party. Under what conditions will someone use the tub?:

(a) Alice never bathes

(b) Bill bathes, iff both Alice and Don are present

(c) Cindy bathes under all conditions - even by herself!

(d) Don bathes iff Alice is not present.

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A = Alice is present, B = Bill is present, C = Cindy is present, D = Don is present, T = someone will use the tub.

T = BAD + C + DA'

= ?

Word Statement: The hot-tub will be used if

Fundamental Theorems & Postulates of Boolean Algebra

Identities:	(1)	X + 0 = X	(1D)	X • 1 = X
Null Elements:	(2)	X + 1 = 1	(2D)	X • 0 = 0
Indempotency:	(3)	X + X = X	(3D)	X • X = X
Involution (Double Negation):			(4)	(X')' = X
Complements:	(5)	X + X' = 1	(5D)	X • X' = 0
Commutativity:	(6)	X + Y = Y + X	(6D)	X • Y = Y • X
Associativity:	(7)	(X+Y)+Z = X+(Y+Z)	(7D)	(X • Y) • Z = X • (Y • Z)
Distributivity:	(8)	X • Y+X • Z = X • (Y+Z)	(8D)	(X+Y) • (X+Z) = X+Y • Z
Combining:	(9)	X • Y + X • Y' = X	(9D)	(X+Y) • (X+Y') = X
Covering:	(10)	X + X • Y = X	(10D)	X • (X+Y) = X
DeMorgan's Laws:	(12)	(X • Y • Z)' = X'+Y'+Z'	(12D)	(X+Y+Z)' = X' • Y' • Z'
Consensus:	(17)	X • Y+X' • Z+Y • Z = X • Y+X' • Z	(17D)	(X+Y) • (X'+Z) • (Y+Z) = (X+Y) • (X'+Z)

Shannon Expansion: (18) F(X,Y,Z) = X • F(1,Y,Z) + X' • F(0,Y,Z)

(18D) F(X,Y,Z) = (X+F(0,Y,Z)) • (X'+F(1,Y,Z))

Design Example: The Hot-tub

Reduce using fundamental theorems & postulates (label equation on right side from table)

T = BAD + C + DA'

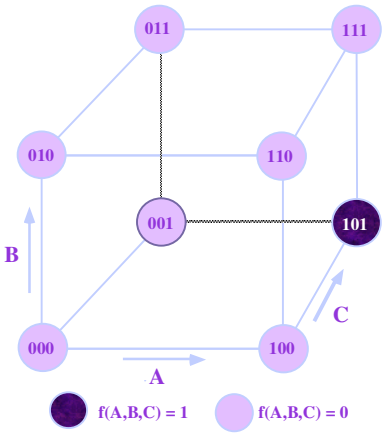
=

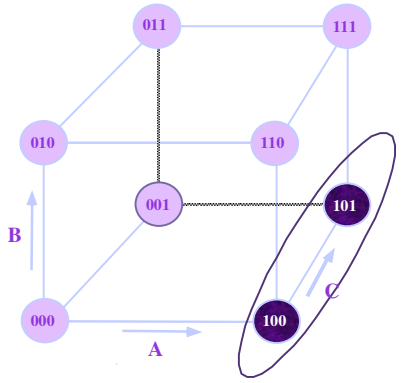
=

=

Word statement: The tub should be warmed up iff:

Cubes as Product Terms

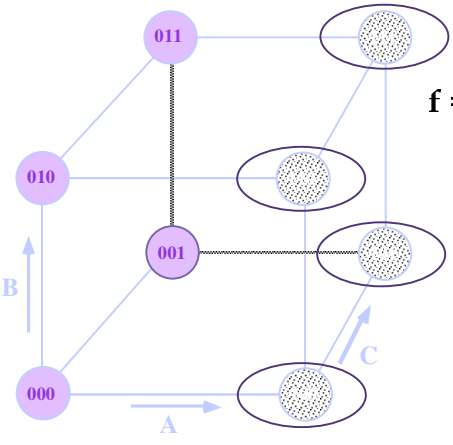




f = AB'C

f = AB'C + AB'C'
= AB'

f (A,B,C) = A
Using Minterms (4 cubes)



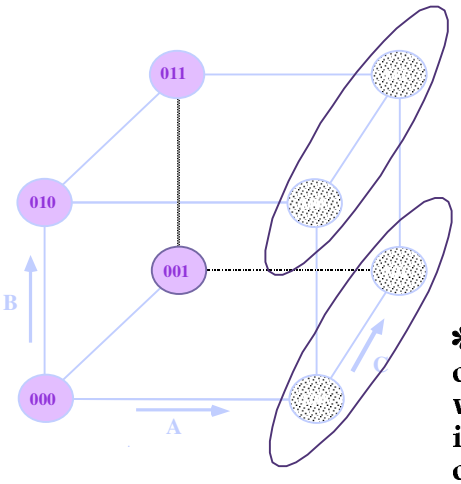
f = AB'C' + AB'C + ABC' + ABC

Alternate format:

A	B	C	f
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

ON-set
or ON-cover

f(A,B,C) = A
Two cubes



f = AB + AB'

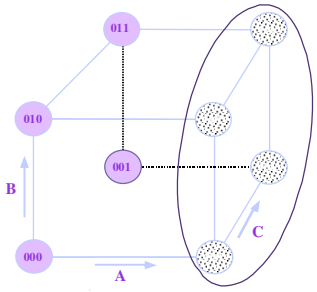
Alternate format:

A	B	C	f
1	1	*	1
1	0	*	1

Input "don't cares"

*** = "don't care": The cube is independent of that variable; we "don't care" what it's value is - it doesn't matter c.f. PLA "don't connects" from last week!**

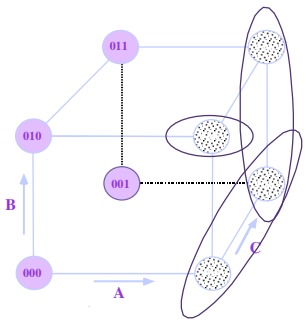
f(A,B,C) = A
One Cube and Mixed Cover



f = A

Alternate format:

A	B	C	f
1	*	*	1

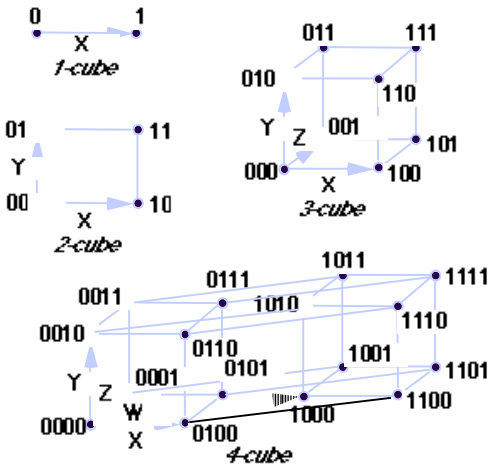


f = ?

Alternate format:

A	B	C	f
1	1	0	1
1	0	*	1
1	*	1	1

Boolean Hypercubes

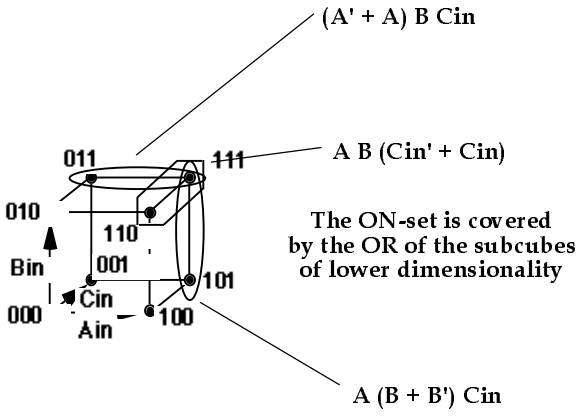


Just another way to represent the truth table

n input variables = n dimensional "cube"

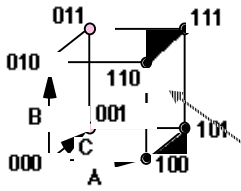
Relationship to Cube Notation: Full Adder

Ain	Bin	Cin	Cout
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



$$Cout = B\,Cin + A\,B + A\,Cin$$

Sub-cubes of Higher Dimensionality



f(A,B,C) = Σm(4,5,6,7)

On-set forms a rectangle,
i.e., a cube of two dimensions

represents an expression in one variable
i.e., 3 dimensions - 2 dimensions

A is asserted and unchanged
B and C vary

This subcube represents the literal A

Describing Logic Functions

- **Definitions:**
  - Adjacent pairs, quadruples, octuples, ... of minterms (a vertex marked 1) in the Boolean space are called *implicants.*
  - An implicant is called a *prime implicant* if it is not a subset of another implicant of the function.
  - Any prime implicant which includes a 1 vertex that is not included in any other prime implicant is called an *essential prime implicant*.
  - A set of prime implicants of a function is called a *minimal sum-of-products form (minimal "cover")* of the function if the following conditions are satisfied:
    - (a) It includes the entire on-set (1 verticies) of the function.
    - (b) Condition (a) will not be satisfied if any one of the prime implicants is removed.
    - (c) None of the prime implicants can be replaced with a simpler prime implicant (i.e. one containing fewer variables).