Detecting and Correcting Bit Errors

Some things we do in computer systems puts the integrity of data at risk.

- Storing bits as charge on capacitors (DRAM, dynamic logic).
- Storing bits with mechanical systems (hard disk, floppy disk).
- Sending bits long distances (copper, fiber, wireless networks).

Today's Lecture

How can we design computer systems to work in the face of bit errors?

We consider digital issues only – analog techniques important, but outside of the scope of this course.

The Big Idea: Redundant Encoding

When storing or sending information, add extra bits (called redundancy).

Error Detection

- When accessing or receiving information, use the extra bits to deduce if corruption occurred.
- Trivial example: send each bit twice, if the values don't match, there was corruption.

Error Correction

- When accessing or receiving information, use the extra bits to fix errors in the data.
- Trivial example: send each bit three times, use a voting scheme to correct single bit errors.
- All error correction has its limits. Trivial example: what if noise corrupted two of the three copies?

Detection Versus Correction

What are the tradeoffs between detection and correction?

Error Detection

+ Needs fewer redundancy bits.

– If an error is detected, receiver requests a resend of the data. This adds latency and complexity.
– If information was destroyed (like a DRAM flipping a bit) there is no data to resend.

Error Correction

+ Low latency, correction happens as a part of the logic for receiving the data.

– Uses more redundancy bits.
– If too many errors happened, data cannot be corrected, and a resend will be needed anyways.

Today's Lecture

Detection and correction implementations.

Part I. Simple Algorithms

- Parity: Single bit detection.
- Hamming Codes: Simple error correction.

Part II. Detecting Multiple Errors

- Theory: Modulo 2 arithmetic
- Practice: Cyclic redundancy codes

Simple Error Detection

Problem: Send an N bit quantity $b_o \ldots b_{n-1}$. Give the receiver a way to detect a single bit error.

Solution: Parity

- Send N+1 bits ($b_o \ldots b_n$).
- Bit b_n is one if there is an odd number of ones amongst $b_o \ldots b_{n-1}$.
- The extra bit is called the **parity bit**.

Computing Parity Bit

- $b_n = b_o \oplus b_1 \oplus \ldots b_{n-1}$
- Where $\oplus$ is the XOR function.
- Parallel data: combinatorial XOR tree.
- Serial data: XOR gate and register bit.

Checking Parity

- $Y = b_o \oplus b_1 \oplus \ldots b_n$
- If $Y == 1$ single-bit error detected.
- Do something sensible if error detected.

When to Use Parity

- Is failure mechanism a good fit for parity?
- Good example: memory array, each b_k on a different chip.
- Bad example: network port that sends all zeros if power fails!

Simple Error Correction

Problem: Send an N bit quantity $b_o \ldots b_{n-1}$. Give the receiver a way to detect and correct a single-bit error.

Hamming Codes

- Take overlapping subsets of $b_o \ldots b_{n-1}$.
- Compute one parity bit for each subset.
- Send M bits: N data bits and K parity bits.
- With enough parity bits, you can ID a flipped data bit.
- Receiver computes a checkword that is 0 (no errors) or whose value is the bit position of the error.
- Complement the bit position of the error to fix.
- R. Hamming, 1950.

Hamming Codes in Detail

Definitions

- N data bits $b_o \ldots b_{n-1}$.
- K parity bits $p_o \ldots p_{k-1}$.
- M total bits $c_1 \ldots c_M$.
- The c's are numbered starting from **one**.

Step 1: Assign b_i's and p_i's to c_i

- Assign each b_i and p_i only **once**.
- First assign c_1, then c_2, etc.
- Assign c_i to a parity bit if i a power of two.
- Elsewise assign c_i to a data bit.
- Stop when no more data bits left to assign.

Assignment Example

Given $N = 4$**:** $b_o \ldots b_3$

$c_1 = p_o$. $(2^0 == 1)$.

$c_2 = p_1$. $(2^1 == 2)$.

$c_3 = b_o$.

$c_4 = p_3$. $(2^2 == 4)$.

$c_5 = b_1$.

$c_6 = b_2$.

$c_7 = b_3$. (last b_i used)

Observations

- $N = 4, K = 3, M = 4 + 3 = 7$
- For large N, $K \approx log_2(N)$.
- This is a good thing.

Step 2: Compute Parity Bits

Recall from introduction . . .

- Take overlapping subsets of the bits.
- Compute one parity bit for each subset.

The questions:

- How to take the subsets from $c_1 \ldots c_M$?

Algorithm:

- Write c subscripts in binary.
- Subsets have a 1 in a particular bit position.
- Each subset has only one p_i.
- Set p_i to XOR of subset data bits.

Parity Computation Example

Recall M = 7

$c_1 = c_{001} = p_0$
$c_2 = c_{010} = p_1$
$c_3 = c_{011} = b_o$
$c_4 = c_{100} = p_3$
$c_5 = c_{101} = b_1$
$c_6 = c_{110} = b_2$
$c_7 = c_{111} = b_3$

Subsets

$(c_4, c_5, c_6, c_7) == (p_3, b_1, b_2, b_3)$

$(c_2, c_3, c_6, c_7) == (p_1, b_o, b_2, b_3)$

$(c_1, c_3, c_5, c_7) == (p_0, b_o, b_1, b_3)$

To Compute Parity

- XOR data bits of the subset.
- Assign to parity bit of the subset.
- Each subset has one parity bit by design.

Step 3: Correct Errors on Reception

Recall from introduction ...

- With enough parity bits, you can ID a flipped data bit.
- Receiver computes a checkword that is 0 (no errors) or whose value is the bit position of the error.

The Question:

- How to compute the checkword?

Algorithm:

- Break data into subsets.
- If subset contains p_i, set w_i to XOR of all subset members.
- Checkword $= w_0 + 2 * w_1 + 4 * w_2 + \ldots$
- If checkword is zero, no errors.
- If checkword is nonzero, flip bit $c_{\text{checkword}}$.

Checkword Computation Example

$$\begin{aligned}
c_1 &= c_{001} = p_0 \\
c_2 &= c_{010} = p_1 \\
c_3 &= c_{011} = b_o \\
c_4 &= c_{100} = p_3 \\
c_5 &= c_{101} = b_1 \\
c_6 &= c_{110} = b_2 \\
c_7 &= c_{111} = b_3
\end{aligned}$$

Subsets

$$(c_4, c_5, c_6, c_7) == (p_3, b_1, b_2, b_3)$$

$$(c_2, c_3, c_6, c_7) == (p_1, b_o, b_2, b_3)$$

$$(c_1, c_3, c_5, c_7) == (p_0, b_o, b_1, b_3)$$

To Compute Checkword

$$w_2 = c_4 \oplus c_5 \oplus c_6 \oplus c_7 = p_3 \oplus b_1 \oplus b_2 \oplus b_3$$

$$w_1 = c_2 \oplus c_3 \oplus c_6 \oplus c_7 = p_1 \oplus b_o \oplus b_2 \oplus b_3$$

$$w_0 = c_1 \oplus c_3 \oplus c_5 \oplus c_7 = p_0 \oplus b_o \oplus b_1 \oplus b_3$$

$$\text{Checkword} = w_0 + 2 * w_1 + 4 * w_2$$

Hamming Code Epilogue

Why Does It Work?

- Uses binary code to advantage.
- If checkword not zero, a bit must be wrong.
- Each $w_i = 0$ removes some bits from suspicion.
- Whichever bit is left must be wrong.
- If a bit is wrong, only one way to make it right!

Help! I'm Confused!

- Its normal to be confused after seeing it just once.
- Print these slides, reread.

Detecting Multiple Errors

For Use When:

- Sending packets of data over an unreliable medium.
- Most of the time, bits arrive perfectly.
- Sometimes, many bits are corrupted.
- Need to detect corruption, request a resend.
- Parity too weak: multi-bit corruption is common.

Example Applications:

- Ethernet Packets
- Internet Protocol Datagrams

Basic Idea: Checksums

- Given a data packet of N bits.
- Compute an M bit **checksum** word on data
- Send M checksum bits and N data bits.
- Receiver also computes checksum on packet.
- Packet corrupt if checksums are different.
- Parity is a very weak 1-bit checksum.

Desirable Checksum Properties

- N may vary packet by packet, M is fixed.
- Algorithm should be easy to compute.
- Algorithm and M chosen to match application.

Example: Cyclic Redundancy Check (CRC)

- Used in Ethernet Packets (M = 32).
- Ethernet P(undetected error) = $1/2^{32}$
- Less than one in a billion.

The Basic Idea

- Treat data packet as an N bit number.
- Divide number by a constant.
- The "remainder" of the division is the checksum.

But Division is Slow!

- Solution: use a number system where division is fast.
- Modulo 2 Arithmetic.

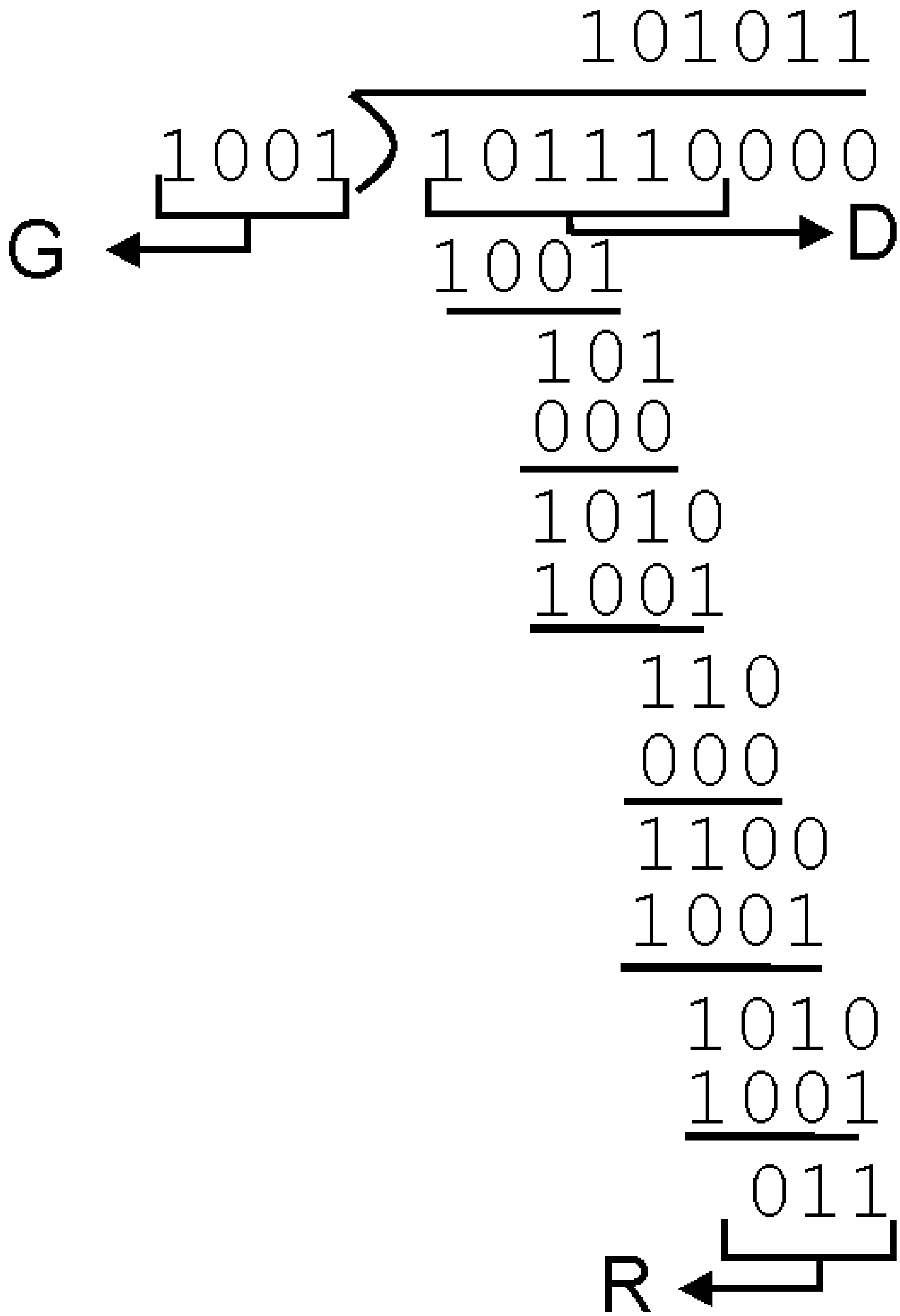
Modulo 2 Arithmetic

- + is bit-wise XOR (no carries).
- - is bit-wise XOR (no borrows).
- * is bit-wise AND (no carries).
- Division uses - and * above (no borrows).

Not an Ad-hoc Scheme!

- Operations define a finite (Galois) field.
- Associative and distributive properties.
- + and * identities and inverses.
- Ops map one N bit word to another N bit word.

Modulo 2 Division



CRC Algorithm In Detail

- D is the N-bit data word to be sent.
- We compute M-bit checksum word R.
- Send concatenation: $2^M * D + R$

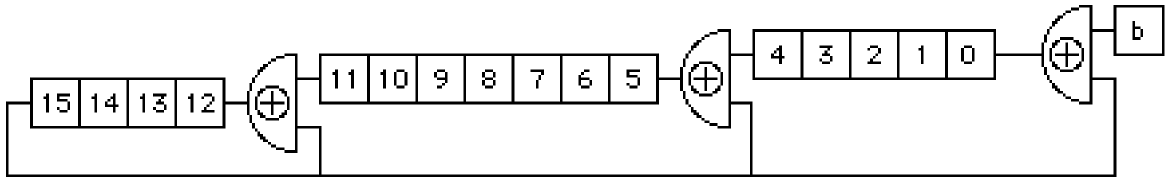
Computing Checksum

- Compute $(2^M * D)/G = Q + R/G$
- Ethernet $G =$ 100000100110000010001110110110111
- Checksum is remainder R.

On Receipt

- Divide $(2^M * D + R)$ by G
- $(2^M * D)/G + (R/G)$
- From above: $Q + (R/G) + (R/G)$
- Rewrite: $Q + (R/G) - (R/G)$
- Remainder should be zero! If not, corrupt.

CRC Hardware



- Serial dividend flows from b box.
- Flip-flops hold 16-bit remainder R.
- 17-bit divisor $G = 10001000000100001$
- Input XOR is first and last 1.
- Two middle 1's map to middle XORs.

Preparing Checksum

- Clear flip-flops.
- Append 16 zeros to N-bit data D
- $16 + N$ clocks to enter data.
- Remainder R sits in flip-flops.

Upon Receipt of $2^{16} * D + R$

- Clear flip-flops.
- $16 + N$ clocks to enter data.
- A non-zero flip-flop flags an error.

How Does It Work?

- The secret: it doesn't work for all divisors.
- Only for numbers with "cyclic" property.
- Theoretical background needed to go further.
- Linear Feedback Shift Registers
- Also used for random number generation.

In Conclusion

- Error coding is where math meets logic.
- Modulo 2 bit-serial circuits are powerful.
- Deep mathematics behind the gates.